Interactive Formal Verification
1: Introduction

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Motivation

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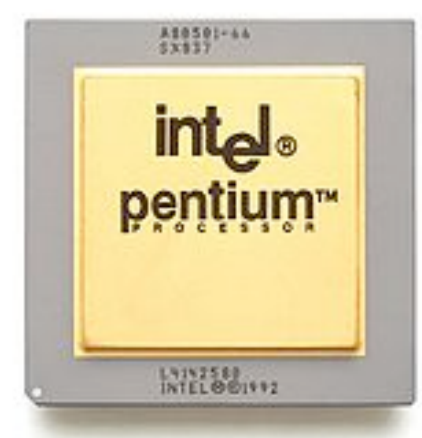
- Program testing can be used to show the presence of bugs, but never to show their absence!

  Edsger W. Dijkstra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- Work in a logical formalism
  - precise definitions of concepts
  - formal reasoning system
- Construct hierarchies of definitions and proofs
  - libraries of formal mathematics
  - specifications of components and properties

Interactive Theorem Provers

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- Based on first-order logic with recursion
  - ACL2

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- ... but the implementation is more complicated, and performance can suffer.
- Used in Isabelle, HOL, Coq but not PVS or ACL2.

Theorem Provers: Characteristic Features

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- Tools

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- Integrated tool support for
    - Automated provers
    - Counterexamples
    - Code generation
    - LaTeX document generation

Higher-Order Logic

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- Polymorphic types, including a type of truth values
- No distinction between terms and formulas
- ML-style functional programming

"HOL = functional programming + logic"

Basic Syntax of Formulas

formulas *A*, *B*, ... can be written as

`(`A`)`	`t = u`	`~`A
A `&` B	A `|` B	A `-->` B
A `<->` B	`ALL` $x.A$	`EX` $x.A$

(Among many others)

Isabelle also supports symbols such as

$\leq \ \geq \ \neq \ \wedge \ \vee \ \rightarrow \ \leftrightarrow \ \forall \ \exists$

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Binary logical connectives associate to the right: $A \rightarrow B \rightarrow C$ is the same as $A \rightarrow (B \rightarrow C)$

$\neg A \wedge B = C \vee D$ is the same as $((\neg A) \wedge (B = C)) \vee D$

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  - constants, c
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  - abstractions λx. t
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- The typed λ-calculus:
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  - variables, x and *flexible* variables, ?x
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  - function applications t u
- Numerous infix operators and binding operators for arithmetic, set theory, etc.

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- There are types of ordered pairs and functions.
- Other important types are those of the natural numbers (`nat`) and integers (`int`).

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- Extensible record types can also be defined.

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  - Curried function notation: $\lambda x\ y.\ t$
- Function arguments can be paired
  - Example: `nat*nat => nat`
  - Paired function notation: $\lambda(x,y).\ t$

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  - inductively defined: `0`, `Suc` ***n***
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- `rat`, `real`: `+ - * / sin cos ln` ...
- arithmetic constants and laws for these types

HOL as a Functional Language

recursive data type of lists

```
datatype 'a list = Nil | Cons 'a "'a list"

fun app :: "'a list => 'a list => 'a list"  where
   "app Nil ys = ys"
| "app (Cons x xs) ys = Cons x (app xs ys)"

fun rev  where
   "rev Nil = Nil"
| "rev (Cons x xs) = app (rev xs) (Cons x Nil)"
```

recursive functions
(types can be inferred)

Proof by Induction

declaring a lemma

use it to simplify other formulas

```
lemma [simp]: "app xs Nil = xs"
  apply (induct xs)
  apply auto
  done
```

two steps: *induction* followed by *automation*

end of proof

Example of a *Structured Proof*

```
lemma "app xs Nil = xs"
proof (induct xs)
  case Nil
  show "app Nil Nil = Nil"
    by auto
next
  case (Cons a xs)
  show "app (Cons a xs) Nil = Cons a xs"
    by auto
qed
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Example of a *Structured Proof*

- base case and inductive step can be proved explicitly

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- Invaluable for proofs that need intricate manipulation of facts

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